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NO $N = 4$ STRINGS ON WOLF SPACES ¹

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Abstract

We generalize the standard $N = 2$ supersymmetric Kazama-Suzuki coset construction to the $N = 4$ case by requiring the *non-linear* (Goddard-Schwimmer) $N = 4$ quasi-superconformal algebra to be realized on cosets. The constraints that we find allow very simple geometrical interpretation and have the Wolf spaces as their natural solutions. Our results obtained by using components-level superconformal field theory methods are fully consistent with standard results about $N = 4$ supersymmetric two-dimensional non-linear sigma-models and $N = 4$ WZNW models on Wolf spaces. We construct the actions for the latter and express the quaternionic structure, appearing in the $N = 4$ coset solution, in terms of the symplectic structure associated with the underlying Freudenthal triple system. Next, we gauge the $N = 4$ QSCA and build a quantum BRST charge for the $N = 4$ string propagating on a Wolf space. Surprisingly, the BRST charge nilpotency conditions rule out the non-trivial Wolf spaces as consistent string backgrounds.

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1 Introduction

The critical (non-topological)³ $N = 4$ strings are known since 1976 [1], but they received little attention in the literature because of their apparently ‘negative’ critical dimension. By the critical dimension one actually means the formal number of irreducible 2d scalar $N = 4$ multiplets whose contribution to the conformal anomaly cancels the contribution of $N = 4$ ghosts that arise in gauge-fixing the $N = 4$ superconformal supergravity multiplet. A closer inspection of the argument reveals at least two relevant things: (i) it is implicit that the $N = 4$ string constraints have to form the ‘*small*’ linear $N = 4$ *superconformal algebra* (SCA) having the $\widehat{su(2)}$ affine Lie subalgebra, and (ii) the background space in which such $N = 4$ strings are supposed to propagate is *flat*.

In this paper, we are going to challenge both assumptions in an attempt to find new consistent $N = 4$ string theories. First of all, we replace the ‘small’ linear $N = 4$ SCA by the more general non-linear $N = 4$ *quasi-superconformal algebra* (QSCA) found by Goddard and Schwimmer [2] and closely related with the ‘large’ linear $N = 4$ SCA, having two affine $\widehat{su(2)}$ subalgebras. Second, we choose a *coset* G/H as the embedding space. The embedding space should be general enough to accommodate as much as possible representations of the underlying QSCA, but not to be too general in order to still allow an explicit treatment. Cosets perfectly satisfy both requirements, as is well known in *(super)conformal field theory* (SCFT). Requiring $N = 4$ supersymmetry severely constrains the cosets in question, and it is one of our main purposes to determine which cosets are compatible with the $N = 4$ non-linear QSCA.

We first generalize the standard $N = 0, 1$ *Goddard-Kent-Olive* (GKO) [3] and $N = 2$ *Kazama-Suzuki* (KS) [4] coset constructions to the $N = 4$ case (secs. 2 and 3). Next, we require the $N = 4$ supersymmetry in the general 2d non-linear scalar field theory and in the *Wess-Zumino-Witten-Novikov* (WZNW) models (sect. 4), which complements the $N = 4$ SCFT construction of sect. 3. As far as the *linear* $N = 4$ SCA’s are concerned, Sevrin and Theodoridis [5] found an $N = 4$ generalization of the GKO and KS coset constructions in SCFT by imposing the ‘large’ linear $N = 4$ SCA in $N = 1$ superspace. They found coset solutions of the type $W \otimes SU(2) \otimes U(1)$, where W is a Wolf space. We take a different approach by requiring a coset to support the non-linear $N = 4$ QSCA, and using components. Our constraints allow very simple geometrical interpretation, and have just the Wolf spaces as their solutions. Our SCFT results are perfectly consistent

³By non-topological strings we mean strings based on untwisted *two-dimensional* (2d) (super)conformal algebras, with the usual relation between spin and statistics.

with the standard results about the 2d *non-linear sigma-models* (NLSM's) with N -extended supersymmetry. To solve our $N = 4$ constraints completely, we provide their alternative derivation, by constructing the relevant $N = 4$ WZNW models on Wolf spaces. Based on the triple system construction of the N -extended SCA's developed by Günaydin [6], we express the quaternionic structure, appearing in the $N = 4$ coset solution, in terms of the symplectic structure associated with the underlying *Freudenthal triple system* (FTS). Next, we promote the symmetry realized by the $N = 4$ QSCA to the local level in order to get the corresponding $N = 4$ string, and build the string BRST charge. Requiring its nilpotency is shown to lead to severe constraints on the cosets in question. Finally, we briefly discuss a connection to the known results [7, 8] about the on- and off-shell structure of matter couplings in extended supergravities in four and two dimensions (sect. 5). Our conclusion and outlook are summarized in sect. 6. The defining equations of the $N = 4$ QSCA are collected in Appendix.

2 Supersymmetric Coset Constructions

In this section we review some well-known standard constructions in 2d SCFT, including the KS construction for $N = 2$. This gives the necessary pre-requisite for the $N = 4$ SCFT coset construction to be discussed in the next section, and introduces our notation.

2.1 Affine Lie algebras and Sugawara construction

Let \mathcal{G} be the Lie algebra associated with a semi-simple Lie group G , and f^{abc} and $|G|$ be its structure constants and dimension, respectively, $a, b = 1, 2, \dots, |G|$. Given a non-trivial representation $t_{(r)}^a$ of \mathcal{G} , let us consider the trace, $\text{tr} \left(t_{(r)}^a t_{(r)}^b \right) \equiv g_{(r)}^{ab}$, defining the normalization metric $g_{(r)}^{ab}$. This metric can always be diagonalized in the representation space (\mathcal{G} -module),

$$\text{tr} \left(t_{(r)}^a t_{(r)}^b \right) = l_r \delta^{ab} . \quad (2.1)$$

In particular, as far as the adjoint (A) representation is concerned, the metric g_A^{ab} is known as the Cartan-Killing metric, and its canonical form is given by

$$f^{acd} f^{bdc} = l_A \delta^{ab} . \quad (2.2)$$

The Casimir eigenvalue C_r associated with representation $t_{(r)}^a$ is defined by

$$C_r \delta^{\alpha\beta} = \sum_a \left(t_{(r)}^a t_{(r)}^a \right)^{\alpha\beta} . \quad (2.3)$$

Eqs. (2.1) and (2.3) imply the relation $C_r d_r = l_r |G|$, where the dimension d_r of representation (r) has been introduced, $\alpha, \beta = 1, 2, \dots, d_r$. The normalization of representation (r) is therefore fixed by the coefficient l_r alone. If the sum in eq. (2.1) were restricted to the Cartan subalgebra of \mathcal{G} , we would get instead

$$\sum_{k=1}^{d_r} \mu_{(k)}^2 = l_r r_G, \quad (2.4a)$$

where r_G is the rank of the group G , and μ are the weights of the representation (r) . In particular, as far as the adjoint representation is concerned, we have $d_A = |G|$ and

$$C_A = l_A = r_G^{-1} \sum_{a=1}^{|G|} \alpha_{(a)}^2, \quad (2.4b)$$

where α 's are the roots of \mathcal{G} . Let ψ be the highest root. Then the normalization-independent quantity

$$\tilde{h}_G \equiv C_A / \psi^2 = \frac{1}{r_G} \left[n_L + \left(\frac{S}{L} \right)^2 n_S \right], \quad (2.5)$$

where n_L and n_S denote the numbers of long and short roots, respectively, is known as the dual Coxeter number. The roots in classical Lie algebras are known to come in two lengths at most. The Dynkin diagrams having only single lines have roots all of the same length, and they correspond to the so-called simply-laced Lie algebras.

Let $J^a(z)$ be generators for the associated affine Lie algebra $\hat{\mathcal{G}}$ of level k_G ,

$$J^a(z) J^b(w) \sim \frac{\delta^{ab} k_G / 2}{(z-w)^2} + \frac{i f^{abc}}{z-w} J^c(w). \quad (2.6)$$

The Sugawara stress tensor is defined by ⁴

$$T(z) = \frac{1}{k_G + \tilde{h}_G} \sum_{a=1}^{|G|} J^a(z) J^a(z), \quad (2.7)$$

and it has central charge

$$c_G = \frac{k_G |G|}{k_G + \tilde{h}_G}. \quad (2.8)$$

One can think of this CFT construction as realized by the 2d WZNW theory based on the group G (see sect. 4 for more). As is well known, the level k_G must be a positive integer for unitary affine representations, as well as for the WZNW action to be well-defined.

⁴Normal ordering is implicit in our formulae.

2.2 Super-affine Lie algebras and the associated super-Virasoro algebras

The WZNW theory is the particular 2d non-linear sigma-model (with WZ torsion) on a group manifold, and it can be made ($N = 1$) supersymmetric along the standard lines, either in components or in superspace. It follows that the WZNW fermions, which are the superpartners of the WZNW bosons (in the adjoint representation), are actually *free* fields (sect. 4). This can be understood by noticing that the WZNW fields take their values in a group manifold with the *parallelizing* torsion represented by the WZ term and, hence, the spin connection present in the Lie algebra-valued covariant derivative acting on the WZNW fermions should be trivial.

Let $\psi^a(z)$ be a set of (holomorphic) free fermions in the adjoint representation, which can be thought of as originated from the super-WZNW theory, with the canonical OPE's

$$\psi^a(z)\psi^b(w) \sim -\frac{\delta^{ab}}{z-w} . \quad (2.9)$$

One can always associate affine currents with free fermions,

$$J_f^a(z) = \frac{i}{2} f^{abc} \psi^b(z) \psi^c(z) , \quad (2.10)$$

which define a representation of $\widehat{\mathcal{G}}$ at level $k_G = \tilde{h}_G$. The Sugawara construction for free fermions in the adjoint representation gives the stress tensor which is equivalent to the usual free (quadratic in the fields) fermionic stress tensor, and it has the central charge $c_f = \frac{1}{2}|G|$, as it should.

This is to be compared with the defining OPE's of an $N = 1$ supersymmetric affine Lie algebra,

$$\begin{aligned} J^a(z)J^b(w) &\sim \frac{\delta^{ab}k_G/2}{(z-w)^2} + \frac{if^{abc}J^c(w)}{z-w} , \\ J^a(z)j^b(w) &\sim \frac{if^{abc}j^c(w)}{z-w} , \\ j^a(z)j^b(w) &\sim \frac{\delta^{ab}k_G/2}{z-w} . \end{aligned} \quad (2.11)$$

Defining

$$j^a(z) = j_f^a(z) \equiv i\sqrt{\tilde{h}_G/2} \psi^a(z) , \quad (2.12a)$$

and

$$J^a(z) = J_f^a(z) \equiv -\frac{i}{\tilde{h}_G} f^{abc} j^b(z) j^c(z) , \quad (2.12b)$$

we therefore obtain the free-fermionic representation of the super-affine Lie algebra at the level $k_G = \tilde{h}_G$. Similarly, the Sugawara bosonic construction can also be supersymmetrized to the full $N = 1$ super-Virasoro algebra by introducing a dimension-3/2 current $G_f(z)$ which is the superpartner of $T_f(z)$. The supercurrent G_f must square to T_f , and its explicit form is given by

$$G_f(z) = -\frac{1}{3\sqrt{2\tilde{h}_G}} f^{abc} \psi^a \psi^b \psi^c . \quad (2.13)$$

One easily finds

$$\begin{aligned} G_f(z) J_f^a(w) &\sim \frac{1}{(z-w)^2} j_f^a(z) , \\ G_f(z) j_f^a(w) &\sim \frac{1}{z-w} J_f^a(w) . \end{aligned} \quad (2.14)$$

Of course, all the above-mentioned is valid for *any* free fermions, not just for those belonging to the super-WZNW theory. If, nevertheless, our free fermions originate from the super-WZNW theory, we still have at our disposal the bosonic currents $\hat{J}^a(z)$ forming a level- k_G representation of affine Lie algebra $\hat{\mathcal{G}}$, which are independent on the fermionic fields. We are therefore in a position to define general affine representations,

$$J^a(z) = J_f^a(z) + \hat{J}^a(z) , \quad (2.15)$$

of level

$$k = k_G + \tilde{h}_G , \quad (2.16)$$

and of central charge

$$c = \frac{k_G |G|}{k_G + \tilde{h}_G} + \frac{1}{2} |G| . \quad (2.17)$$

It can be extended to a representation of the super-affine algebra by adding

$$j^a(z) = i\sqrt{k/2} \psi^a(z) . \quad (2.18)$$

The (Sugawara-type) $N = 1$ super-Virasoro algebra associated with this construction is given by

$$\begin{aligned} T(z) &= \frac{1}{k} \left[\hat{J}^a(z) \hat{J}^a(z) - j^a(z) \partial j^a(z) \right] , \\ G(z) &= \frac{2}{k} \left[j^a(z) \hat{J}^a(z) - \frac{i}{3k} f^{abc} j^a(z) j^b(z) j^c(z) \right] . \end{aligned} \quad (2.19)$$

The 2d field theory realization of this CFT construction is provided by the quantized super-WZNW theories (sect. 4).

2.3 Coset (GKO) constructions

A much larger class of (S)CFT's can be obtained by the coset method, also known as the GKO construction. It was even conjectured that coset models may exhaust all *rational* conformal field theories (RCFT's). Let H be a subgroup of G , \mathcal{H} the Lie algebra of H , and $J_H^i(z)$ the affine $\hat{\mathcal{H}}$ -currents, $i, j = 1, 2, \dots, |H|$. We assume that the first $|H|$ currents in $\{J_G^a\}$ just represent the currents $\{J_H^i\}$. As far as our notation is concerned, *early* lower case Latin indices are used for G -indices, *middle* lower case Latin indices are used for H -indices, while early lower case Latin indices *with bars* are used for G/H -indices, $a = (i, \bar{a})$ and $\bar{a} = |H| + 1, \dots, |G|$. We have

$$\begin{aligned} J_G^a(z) J_G^b(w) &\sim \frac{\delta^{ab} k_G / 2}{(z-w)^2} + \frac{i f^{abc}}{z-w} J_G^c(w) , \\ J_H^i(z) J_H^j(w) &\sim \frac{\delta^{ij} k_H / 2}{(z-w)^2} + \frac{i f^{ijk}}{z-w} J_H^k(w) . \end{aligned} \tag{2.20}$$

The level k_H is determined by embedding of \mathcal{H} into \mathcal{G} . An embedding is characterized by the embedding index I_H defined by $I_H = \psi_G^2 / \psi_H^2$ which is always an integer. As far as the bosonic WZNW currents $\{\hat{J}_H\} \subset \{\hat{J}_G\}$ are concerned, we obviously have $k_H = I_H k_G$. In particular, if the simple roots of H form a subset of the simple roots of G , then $I_H = 1$ and $k_H = k_G$.⁵

Having restricted the free fermions ψ^a in the adjoint of G to the subset ψ^i in the adjoint of H , we can introduce the affine currents

$$J_{H,f}^i(z) = \frac{i}{2} f^{ijk} \psi^j(z) \psi^k(z) \tag{2.21}$$

forming a representation of $\hat{\mathcal{H}}$ of level $k_H = \tilde{h}_H$. Still, there is another natural representation of $\hat{\mathcal{H}}$, also associated with the free fermions and defined by the currents

$$J_{G/H,f}^i(z) = \frac{i}{2} f_{\bar{b}\bar{c}}^i \psi^{\bar{b}}(z) \psi^{\bar{c}}(z) \tag{2.22}$$

of level $k_{H,f} = I_H \tilde{h}_G - \tilde{h}_H$, where \tilde{h}_H is the dual Coxeter number for \mathcal{H} . Therefore, after taking into account eqs. (2.15) and (2.16), one finds that $k_H = I_H k_G + I_H \tilde{h}_G - \tilde{h}_H$ in general.

⁵ $I_H = 3$ when $G = G_2$ and $H = SU(2)$, whereas $I_H = 2$ when $G = SO(7)$ and $H = SO(3)$.

The Sugawara stress tensors associated with the affine G and H currents take the form

$$\begin{aligned} T_G(z) &= \frac{1}{k_G + \tilde{h}_G} \sum_{a=1}^{|G|} J_G^a(z) J_G^a(z) , \\ T_H(z) &= \frac{1}{k_H + \tilde{h}_H} \sum_{i=1}^{|H|} J_H^i(z) J_H^i(z) . \end{aligned} \quad (2.23)$$

The corresponding Virasoro central charges are

$$c_G = \frac{k_G |G|}{k_G + \tilde{h}_G} , \quad c_H = \frac{k_H |H|}{k_H + \tilde{h}_H} . \quad (2.24)$$

The standard (GKO) coset construction is defined by

$$T_{G/H} = T_G - T_H , \quad (2.25a)$$

and it has central charge

$$c_{G/H} = c_G - c_H = \frac{k_G |G|}{k_G + \tilde{h}_G} - \frac{k_H |H|}{k_H + \tilde{h}_H} . \quad (2.25b)$$

The $N = 1$ generalization of the GKO construction given above is based on an *orthogonal* decomposition of the $N = 1$ SCA associated with the group G , with respect to its subgroup H ,

$$\begin{aligned} T_G(z) &= T_H(z) + T_{G/H}(z) , \\ G_G(z) &= G_H(z) + G_{G/H}(z) , \end{aligned} \quad (2.26)$$

where H - and G/H - currents are to be mutually commuting. To actually get such a decomposition, one uses two $\hat{\mathcal{H}}$ -representations introduced in eqs. (2.12), (2.15) and (2.22), namely

$$\tilde{J}^i(z) = \hat{J}^i(z) + \frac{i}{2} f^{i\bar{b}\bar{c}} \psi^{\bar{b}}(z) \psi^{\bar{c}}(z) . \quad (2.27)$$

and

$$J^i(z) = \tilde{J}^i(z) + \frac{i}{2} f^{imn} \psi^m(z) \psi^n(z) , \quad (2.28)$$

where $\{\hat{J}_H\} \subset \{\hat{J}_G\}$ are the bosonic currents forming a level- k_H representation of $\hat{\mathcal{H}}$. The stress tensor $T_H(z)$ and the supercurrent $G_H(z)$ are defined by

$$\begin{aligned} T_H(z) &= \frac{1}{k} \tilde{J}^i(z) \tilde{J}^i(z) + \frac{1}{2} \psi^i(z) \partial \psi^i(z) , \\ G_H(z) &= \frac{i}{\sqrt{2k}} \psi^i(z) \tilde{J}^i(z) - \frac{1}{3\sqrt{2k}} f^{imn} \psi^i(z) \psi^m(z) \psi^n(z) , \end{aligned} \quad (2.29)$$

where eq. (2.19) has been used as a guide. Note that the ‘improved’ current \tilde{J}^i instead of the ‘naive’ bosonic current \hat{J}^i has been used in eq. (2.29). This is possible since $\tilde{J}^i(z)$ commutes with $j^i(z)$. Most importantly, eq. (2.26) yields the desired *orthogonal* decomposition since so *defined* $T_{G/H}(z)$ and $G_{G/H}(z)$ commute with $J^i(z)$, $\tilde{J}^i(z)$ and $j^i(z)$.⁶ Explicitly, they read

$$\begin{aligned} T_{G/H}(z) &= \frac{1}{k} \left[\hat{J}^{\bar{a}}(z) \hat{J}^{\bar{a}}(z) + \frac{k_G}{2} \psi^{\bar{a}}(z) \partial \psi^{\bar{a}}(z) - i \hat{J}^i(z) f^{\bar{i}\bar{b}\bar{c}} \psi^{\bar{b}}(z) \psi^{\bar{c}}(z) \right. \\ &\quad \left. + f^{\bar{a}\bar{c}\bar{d}} f^{\bar{b}\bar{c}\bar{d}} \psi^{\bar{a}}(z) \partial \psi^{\bar{b}}(z) - \frac{1}{4} f^{\bar{a}\bar{b}\bar{c}} f^{\bar{a}\bar{g}\bar{f}} \psi^{\bar{b}}(z) \psi^{\bar{c}}(z) \psi^{\bar{g}}(z) \psi^{\bar{f}}(z) \right] , \\ G_{G/H}(z) &= \frac{i}{\sqrt{2k}} \psi^{\bar{a}}(z) \hat{J}^{\bar{a}}(z) - \frac{1}{3\sqrt{2k}} f^{\bar{a}\bar{b}\bar{c}} \psi^{\bar{a}}(z) \psi^{\bar{b}}(z) \psi^{\bar{c}}(z) , \end{aligned} \quad (2.30)$$

and have central charge

$$\begin{aligned} c_{G/H} = c_G - c_H &= \left[\frac{k_G |G|}{k_G + \tilde{h}_G} + \frac{1}{2} |G| \right] - \left[\frac{(I_H k_G + I_H \tilde{h}_G - \tilde{h}_H) |H|}{I_H (k_G + \tilde{h}_G)} + \frac{1}{2} |H| \right] ; \\ c_{G/H} &= \frac{3k_G}{2k} \dim(G/H) + \frac{1}{8k} f^{\bar{a}\bar{b}\bar{c}} f^{\bar{a}\bar{b}\bar{c}} , \quad \text{when } I_H = 1 , \end{aligned} \quad (2.31)$$

where $k = k_G + \tilde{h}_G$ as above. In particular, for a *symmetric* space G/H where $f^{\bar{a}\bar{b}\bar{c}} = 0$, one finds

$$\begin{aligned} T_{G/H}(z) &= \frac{1}{k} \left[\hat{J}^{\bar{a}}(z) \hat{J}^{\bar{a}}(z) + \frac{k_G}{2} \psi^{\bar{a}}(z) \partial \psi^{\bar{a}}(z) - i f^{\bar{i}\bar{b}\bar{c}} \hat{J}^i(z) \psi^{\bar{b}}(z) \psi^{\bar{c}}(z) \right] , \\ G_{G/H}(z) &= \frac{i}{\sqrt{2k}} \psi^{\bar{a}}(z) \hat{J}^{\bar{a}}(z) , \end{aligned} \quad (2.32)$$

and

$$c_{G/H} = c_G - c_H = \frac{3k_G}{2(k_G + \tilde{h}_G)} \dim(G/H) , \quad (2.33)$$

according to eqs. (2.16) and (2.31) with $I_H = 1$.

2.4 KS construction

Having obtained the $N = 1$ super-Virasoro algebra associated with the $N = 1$ super affine Lie algebra, it is quite natural to ask about the conditions on the coset G/H which would allow more supersymmetries, i.e. $N > 1$. The case of $N = 2$ was fully addressed by Kazama and Suzuki [4]. Since the $N = 2$ extended SCA has a second

⁶The generators T_G and G_G have been introduced before, see eq. (2.19).

supercurrent and an abelian $U(1)$ current beyond the content of the $N = 1$ SCA, the $N = 2$ conditions on the coset G/H just originate from requiring their existence. The most general ansatz for the second supercurrent takes the form [4]

$$G^{(2)}(z) = \frac{i}{\sqrt{2k}} h_{\bar{a}\bar{b}} \psi^{\bar{a}}(z) \hat{J}^{\bar{b}}(z) - \frac{1}{3\sqrt{2k}} S^{\bar{a}\bar{b}\bar{c}} \psi^{\bar{a}}(z) \psi^{\bar{b}}(z) \psi^{\bar{c}}(z) , \quad (2.34)$$

where $h_{\bar{a}\bar{b}}$ and $S^{\bar{a}\bar{b}\bar{c}}$ are constants. The supercurrents $G^{(1)} \equiv G_{G/H}$ and $G^{(2)}$ have to satisfy the basic $N = 2$ SCA OPE

$$G^{(i)}(z) G^{(j)}(w) \sim \frac{2\delta^{ij}c/3}{(z-w)^3} + \frac{2i\varepsilon^{ij}J(w)}{(z-w)^2} + \frac{2\delta^{ij}T(w) + i\delta^{ij}\partial J(w)}{z-w} , \quad (2.35)$$

where the $N = 2$ SCA current $J(z)$ has been introduced. It results in the following $N = 2$ conditions [4]:

$$\begin{aligned} (i) \quad & h_{\bar{a}\bar{b}} = -h_{\bar{b}\bar{a}} , \quad h_{\bar{a}\bar{c}}h_{\bar{c}\bar{b}} = -\delta_{\bar{a}\bar{b}} , \\ (ii) \quad & h_{\bar{a}\bar{c}}f_{\bar{c}\bar{b}d} = h_{\bar{b}\bar{c}}f_{\bar{c}\bar{a}d} , \\ (iii) \quad & f_{\bar{a}\bar{b}\bar{c}} = h_{\bar{a}\bar{f}}h_{\bar{b}\bar{g}}f_{\bar{f}\bar{g}\bar{c}} + h_{\bar{b}\bar{f}}h_{\bar{c}\bar{g}}f_{\bar{f}\bar{g}\bar{a}} + h_{\bar{c}\bar{f}}h_{\bar{a}\bar{g}}f_{\bar{f}\bar{g}\bar{b}} , \\ (iv) \quad & S_{\bar{a}\bar{b}\bar{c}} = h_{\bar{a}\bar{f}}h_{\bar{b}\bar{g}}h_{\bar{c}\bar{h}}f_{\bar{f}\bar{g}\bar{h}} , \end{aligned} \quad (2.36)$$

Given these conditions, the $N = 2$ SCA $U(1)$ current reads

$$J(z) = -\frac{i}{\sqrt{2}} h_{\bar{a}\bar{b}} \psi^{\bar{a}}(z) \psi^{\bar{b}}(z) + \frac{1}{k} h_{\bar{c}\bar{d}} f^{\bar{c}\bar{d}\bar{a}} \left[\hat{J}^{\bar{a}}(z) + \frac{i}{2} f^{a\bar{a}\bar{b}} \psi^{\bar{a}}(z) \psi^{\bar{b}}(z) \right] . \quad (2.37)$$

The conditions (2.36) have simple geometrical interpretation, which allows to describe their solutions in full [4]. In particular, the condition (i) just means that $h_{\bar{a}\bar{b}}$ is an *almost complex* structure on a *hermitian* manifold. The condition (ii) implies that the almost complex structure is *covariantly* constant with respect to the connection with torsion to be defined by the structure constants, whereas the condition (iii) means that the almost complex structure is *integrable*, i.e. it is a *complex structure* indeed (the equation (iii) is equivalent to the vanishing condition on the so-called Nijenhuis tensor [9]). The condition (iv) is the defining equation for $S_{\bar{a}\bar{b}\bar{c}}$. The conditions (ii), (iii) and (iv) are trivially satisfied for the *symmetric* spaces having $f^{\bar{a}\bar{b}\bar{c}} = S^{\bar{a}\bar{b}\bar{c}} = 0$. The *hermitian symmetric* spaces therefore represent an important class of solutions to eq. (2.36), and they were extensively studied [4]. A different class of $N = 2$ supersymmetric solutions is given by the *kählerian* coset spaces which are in fact the *only* solutions if $\text{rank } G = \text{rank } H$ [4]. In general, when $\text{rank } G - \text{rank } H = 2n$, $n = 0, 1, 2, \dots$, the coset $G/[H \otimes U(1)^{2n}]$

must be kählerian [4]. Hence, a solution to the $N = 2$ conditions exists for *any hermitian* coset space. Given a Cartan-Weyl decomposition of \mathcal{G} , the complex structure maps the Cartan subalgebra of \mathcal{G} into itself, whereas the generators corresponding to positive (negative) roots are the eigenvectors with the eigenvalues $+i$ ($-i$).

3 $N = 4$ SCFT coset models

The KS construction delivers a large class of $N = 2$ SCFT's by the coset space method. We now wish to identify those of them which actually possess $N = 4$ supersymmetry. Sevrin and Theodoridis [5] already generalized the KS construction to the $N = 4$ case by requiring the existence of the 'large' *linear* $N = 4$ SCA having $D(2, 1; \alpha)$ as projective subalgebra. The $N = 4$ generators are supposed to act on a coset G/H , i.e. they have to commute with the H generators. Our approach to constructing $N = 4$ SCFT's by the coset space method is however different from the one adopted in ref. [5]. We are going to impose the *non-linear* $N = 4$ supersymmetry because it is more general than the linear one represented by the 'large' $N = 4$ SCA. The 'large' linear $N = 4$ SCA is actually *not* a symmetry algebra since it has subcanonical charges represented by four free fermions and one boson. The proper $N = 4$ supersymmetric symmetry algebra having only canonical charges of dimension 2, $3/2$ and 1 was constructed by Goddard and Schwimmer [2], and we are going to call it the $\hat{D}(2, 1; \alpha)$ *quasi-superconformal algebra* (QSCA) [11]. The $N = 4$ QSCA $\hat{D}(2, 1; \alpha)$ is quadratically *non-linearly* generated. Given a SCFT representing the 'large' linear $N = 4$ SCA, one can always realize over there the $\hat{D}(2, 1; \alpha)$ QSCA too, since the generators of the latter can be non-linearly constructed from the generators of the former (see Appendix). The reverse may not be always possible. We should therefore expect more solutions to exist when imposing the $\hat{D}(2, 1; \alpha)$ QSCA instead of the 'large' linear $N = 4$ SCA. In addition, imposing the QSCA seems to be more satisfactory from the viewpoint of $N = 4$ string theory: the most general algebra to be gauged is not the 'large' linear $N = 4$ SCA but the non-linear $\hat{D}(2, 1; \alpha)$ QSCA! ⁷

The $\hat{D}(2, 1; \alpha)$ QSCA comprises stress tensor $T(z)$, four dimension- $3/2$ supercurrents $G^\mu(z)$, and six dimension-1 currents $J^{\mu\nu}(z)$ in the adjoint of $SO(4) \cong SU(2)_+ \otimes SU(2)_-$. The only non-trivial OPE of this QSCA defines an $N = 4$ supersymmetry algebra in the

⁷See Appendix for a review of both algebras.

form

$$\begin{aligned}
G^\mu(z)G^\nu(w) \sim & \frac{4k^+k^-}{(k^++k^-+2)} \frac{\delta^{\mu\nu}}{(z-w)^3} + \frac{2T(w)\delta^{\mu\nu}}{z-w} \\
& - \frac{k^++k^-}{k^++k^-+2} \left[\frac{2J^{\mu\nu}(w)}{(z-w)^2} + \frac{\partial J^{\mu\nu}(w)}{z-w} \right] \\
& + \frac{k^+-k^-}{k^++k^-+2} \varepsilon^{\mu\nu\rho\lambda} \left[\frac{J^{\rho\lambda}(w)}{(z-w)^2} + \frac{\partial J^{\rho\lambda}(w)}{2(z-w)} \right] \\
& - \frac{\varepsilon^{\mu\rho\lambda\zeta} \varepsilon^{\nu\rho\eta\omega}}{2(k^++k^-+2)} : J^{\lambda\zeta} J^{\eta\omega} : (w) ,
\end{aligned} \tag{3.1}$$

where k^+ and k^- are levels of affine Lie algebras associated with $SU(2)_+$ and $SU(2)_-$, respectively. The tensor $J^{\mu\nu}$ comprises two (anti)self-dual $SU(2)$ triplets ($M = 1, 2, 3$)

$$J^{\mu\nu}(z) = (t^{M-})^{\mu\nu} J^{M-}(z) + (t^{M+})^{\mu\nu} J^{M+}(z) , \tag{3.2}$$

where the antisymmetric 4×4 matrices $t^{M\pm}$ satisfy the relations

$$[t^{M\pm}, t^{N\pm}] = -2\varepsilon^{MNP} t^{P\pm} , \quad [t^{M+}, t^{N-}] = 0 , \quad \{t^{M\pm}, t^{N\pm}\} = -2\delta^{MN} . \tag{3.3}$$

The only non-linear $:JJ:(w)$ term on the r.h.s. of eq. (3.1) can be interpreted as the Sugawara stress tensor for the $\widehat{SO}(4)$ currents. It attributes the $N = 4$ ‘improvement’ to the ‘naive’ stress tensor $T(z)$.

Requiring the $N = 4$ QSCA supersymmetry, we expect the KS conditions (2.36) to be satisfied for each supersymmetry separately. This happens to be true indeed (see below). On dimensional grounds, the general ansatz (2.34) is valid for *any* supersymmetry,

$$G^\mu(z) = \frac{i}{\sqrt{2k}} \left[h_{\bar{a}\bar{b}}^\mu \psi^{\bar{a}}(z) \hat{J}^{\bar{b}}(z) + \frac{i}{3} S_{\bar{a}\bar{b}\bar{c}}^\mu \psi^{\bar{a}}(z) \psi^{\bar{b}}(z) \psi^{\bar{c}}(z) \right] , \tag{3.4}$$

where $h_{\bar{a}\bar{b}}^\mu$ and $S_{\bar{a}\bar{b}\bar{c}}^\mu$ are constants, $\mu = 0, 1, 2, 3$. The OPE for a product of the supercurrents (3.4) takes the form

$$\begin{aligned}
G^\mu(z)G^\nu(w) \sim & -\frac{1}{2k} \left\{ \frac{1}{(z-w)^3} \left[-\frac{k_G}{2} h_{\bar{a}\bar{b}}^\mu h_{\bar{a}\bar{b}}^\nu - \frac{1}{9} S_{\bar{a}\bar{b}\bar{c}}^\mu S_{\bar{a}\bar{b}\bar{c}}^\nu \right] \right. \\
& + \frac{1}{(z-w)^2} \left[\frac{k_G}{2} h_{\bar{a}\bar{c}}^\mu h_{\bar{b}\bar{c}}^\nu \psi^{\bar{a}} \psi^{\bar{b}} - i h_{\bar{a}\bar{c}}^\mu h_{\bar{a}\bar{g}}^\nu f^{\bar{c}\bar{g}d} \hat{J}^d + \frac{1}{3} S_{\bar{a}\bar{b}\bar{f}}^\mu S_{\bar{a}\bar{b}\bar{g}}^\nu \psi^{\bar{f}} \psi^{\bar{g}} \right] \\
& + \frac{1}{z-w} \left[-h_{\bar{c}\bar{a}}^\mu h_{\bar{c}\bar{b}}^\nu \hat{J}^{\bar{a}} \hat{J}^{\bar{b}} + i h_{\bar{a}\bar{c}}^\mu h_{\bar{b}\bar{g}}^\nu f^{\bar{c}\bar{g}d} \hat{J}^d \psi^{\bar{a}} \psi^{\bar{b}} - 2i h_{\bar{a}\bar{b}}^{(\mu} S_{\bar{a}\bar{g}\bar{h}}^{\nu)} \hat{J}^{\bar{b}} \psi^{\bar{g}} \psi^{\bar{h}} \right. \\
& \left. \left. - \frac{k_G}{2} h_{\bar{b}\bar{c}}^\mu h_{\bar{a}\bar{c}}^\nu \psi^{\bar{a}} \partial \psi^{\bar{b}} - \frac{1}{3} S_{\bar{a}\bar{b}\bar{g}}^\mu S_{\bar{a}\bar{b}\bar{f}}^\nu \psi^{\bar{f}} \partial \psi^{\bar{g}} - \frac{1}{3} S_{\bar{a}\bar{b}\bar{c}}^\mu S_{\bar{a}\bar{f}\bar{g}}^\nu \psi^{\bar{b}} \psi^{\bar{c}} \psi^{\bar{f}} \psi^{\bar{g}} \right] \right\} .
\end{aligned} \tag{3.5}$$

Eq. (3.5) is to be compared with eq. (3.1). To get $T = T_{G/H}$ of eq. (2.30) on the r.h.s. of eq. (3.5), let us first look at the coefficients of the terms $(z - w)^{-1} \hat{J} \hat{J}$. This gives the first necessary condition

$$h_{\bar{a}\bar{b}}^\mu h_{\bar{a}\bar{c}}^\nu + h_{\bar{a}\bar{b}}^\nu h_{\bar{a}\bar{c}}^\mu = 2\delta^{\mu\nu} \delta_{\bar{b}\bar{c}} . \quad (3.6)$$

The supercharge $G^0 = G_{G/H}$ of the $N = 1$ subalgebra is defined according to the last line of eq. (2.30), which implies

$$h_{\bar{a}\bar{b}}^0 = \delta_{\bar{a}\bar{b}} , \quad S_{\bar{a}\bar{b}\bar{c}}^0 = f_{\bar{a}\bar{b}\bar{c}} . \quad (3.7)$$

Substituting eq. (3.7) into eq. (3.6) at $\mu = M$ and $\nu = 0$ yields

$$h_{\bar{a}\bar{b}}^M = -h_{\bar{b}\bar{a}}^M , \quad (3.8)$$

whereas taking $\mu = \nu = M$ yields

$$h_{\bar{a}\bar{c}}^M h_{\bar{c}\bar{b}}^M = -\delta_{\bar{a}\bar{b}} , \quad (\text{no sum over } M) . \quad (3.9)$$

Eqs. (3.8) and (3.9) mean that each $h_{\bar{a}\bar{b}}^M$ represents an *almost complex hermitian structure*. Altogether, according to eq. (3.6), they represent an *almost quaternionic tri-hermitian structure*.

The terms of the form $(z - w)^{-1} \hat{J} \psi \psi$ in eq. (3.5) have to deliver the remaining terms in the stress tensor $T_{G/H}$ of eq. (2.30), in particular. We find that this necessarily implies the two conditions:

$$h_{\bar{g}[\bar{a}}^\mu S_{\bar{b}\bar{c}]}^{\nu\bar{g}} + h_{\bar{g}[\bar{a}}^\nu S_{\bar{b}\bar{c}]}^{\mu\bar{g}} = 2\delta^{\mu\nu} f_{\bar{a}\bar{b}\bar{c}} , \quad (3.10)$$

and

$$h_{\bar{a}\bar{b}}^\mu f_{\bar{b}\bar{c}\bar{d}} = h_{\bar{b}\bar{c}}^\mu f_{\bar{a}\bar{b}\bar{d}} . \quad (3.11)$$

Equation (3.10) determines the tensor $S_{\bar{a}\bar{b}\bar{c}}^\mu$ as follows:

$$S_{\bar{a}\bar{b}\bar{c}}^\mu = h_{[\bar{a}}^{\mu\bar{g}} f_{\bar{b}\bar{c}]\bar{g}} , \quad (3.12)$$

or $S_{\bar{a}\bar{b}\bar{c}}^M = h_{[\bar{a}}^{M\bar{g}} f_{\bar{b}\bar{c}]\bar{g}}$. Together with eq. (3.11), it gives us the second consistency condition

$$h_{[\bar{a}}^{\mu\bar{b}} h_{\bar{c}}^{\nu\bar{g}} f_{\bar{d}]\bar{b}\bar{g}} + h_{[\bar{a}}^{\nu\bar{b}} h_{\bar{c}}^{\mu\bar{g}} f_{\bar{d}]\bar{b}\bar{g}} = 2\delta^{\mu\nu} f_{\bar{a}\bar{c}\bar{d}} . \quad (3.13)$$

So far, we only required the relevant stress tensor to appear on the r.h.s. of the supersymmetry algebra in eq. (3.5), which resulted in the necessary conditions (3.6) and

(3.13) for the cosets in question. These equations are also contained in the set of $N = 4$ conditions found by Sevrin and Theodoridis in their work [5]. It is not surprising since they are not sensitive to the differences between the ‘large’ linear $N = 4$ SCA and the non-linear QSCA.⁸ These conditions are therefore very general, and they also have very clear geometrical interpretation [9]. Namely, according to eq. (3.6), there should be three independent almost complex hermitian structures satisfying the quaternionic algebra, thus defining an almost quaternionic tri-hermitian structure on G/H . Eqs. (3.11) and (3.13) guarantee the H -invariance and the covariant constancy of that structure, and imply the vanishing of the Nijenhuis tensor [9]. In other words, the almost quaternionic structure is actually integrable, and defines a quaternionic tri-hermitian structure. The latter appears to be the *only* condition to be satisfied in order that a coset G/H could support $N = 4$ SCFT. All quaternionic manifolds are known to be Einstein spaces of constant non-vanishing scalar curvature. The only known *compact* cases are the Wolf spaces to be discussed below.

Looking at the double-pole terms in eq. (3.5) and comparing them with eq. (3.1), we find the $\widehat{SU(2)}_{\pm}$ currents of the QSCA in the form

$$J^{M-}(z) = \frac{1}{16} h_{\bar{a}\bar{b}}^M \psi^{\bar{a}}(z) \psi^{\bar{b}}(z) , \quad (3.14)$$

and

$$J^{M+}(z) = \frac{i}{4(\tilde{h}_G - 2)} \left[h_{\bar{a}\bar{c}}^M f^{\bar{a}\bar{c}d} \hat{J}^d(z) + \frac{1}{3} h_{\bar{g}[\bar{a}}^M f_{\bar{b}\bar{c}]\bar{g}} f_{\bar{a}\bar{b}} \psi^{\bar{c}}(z) \psi^{\bar{f}}(z) \right] , \quad (3.15)$$

which generalize the results of ref. [10] to non-symmetric spaces. Simultaneously, the levels of the affine Lie subalgebras $\widehat{SU(2)}_{k\pm}$,

$$k^+ = k_G , \quad k^- = \tilde{h}_G - 2 , \quad (3.16)$$

and the $N = 4$ QSCA central charge,

$$c = \frac{6(k_G + 1)(\tilde{h}_G - 1)}{k_G + \tilde{h}_G} - 3 , \quad (3.17)$$

are also fixed. All the generators and the parameters of the non-linear algebra are now determined, and it is straightforward (although quite tedious) to verify the rest of the $\hat{D}(2, 1; \alpha)$ QSCA. No additional consistency conditions arise.

⁸As is shown in the next section, the same conditions follow by requiring the $(1, 0)$ supersymmetric 2d non-linear sigma-model to possess $(4, 0)$ supersymmetry.

As far as the *symmetric* quaternionic spaces are concerned, eqs. (3.4), (3.14), (3.15) and the defining OPE's of the $\hat{D}(2, 1; \alpha)$ algebra in Appendix lead to very simple expressions for the generators of this non-linear algebra on such spaces,

$$\begin{aligned} G^0 &= \frac{i}{\sqrt{2k}} \psi^{\bar{a}} \hat{J}^{\bar{a}} , & G^M &= \frac{i}{\sqrt{2k}} h_{\bar{a}\bar{b}}^M \psi^{\bar{a}} \hat{J}^{\bar{b}} , \\ J^{M-} &= \frac{1}{16} h_{\bar{a}\bar{b}}^M \psi^{\bar{a}} \psi^{\bar{b}} , & J^{M+} &= \frac{i}{4(\tilde{h}_G - 2)} h_{\bar{a}\bar{c}}^M f^{\bar{a}\bar{c}d} \hat{J}^d , \end{aligned} \quad (3.18)$$

$$T = \frac{1}{k} \left[\hat{J}^{\bar{a}} \hat{J}^{\bar{a}} + \frac{1}{2} (k_G + 1) (\psi^{\bar{a}} \partial \psi^{\bar{a}}) - i f^{i\bar{b}\bar{c}} \hat{J}^i \psi^{\bar{b}} \psi^{\bar{c}} + \frac{1}{2} d_{\bar{a}\bar{b}\bar{c}\bar{d}} \psi^{\bar{a}} \psi^{\bar{b}} \psi^{\bar{c}} \psi^{\bar{d}} \right] ,$$

where $d_{\bar{a}\bar{b}\bar{c}\bar{d}}$ are certain linear combinations of the structure constants — see the l.h.s. of eq. (4.17) below.

Given a simple Lie group G , there is the *unique* (associated with this group) quaternionic symmetric space, which is called the *Wolf* space. To introduce this space, let $(E_{\psi\pm}, H_{\psi})$ be the generators of the $su(2)_{\psi}$ subalgebra of \mathcal{G} , associated with the highest root ψ ,

$$[E_{\psi+}, E_{\psi-}] = 2H_{\psi} , \quad [H_{\psi}, E_{\psi\pm}] = \pm E_{\psi\pm} . \quad (3.19)$$

The associated Wolf space is the coset

$$\frac{G}{H_{\perp} \otimes SU(2)_{\psi}} , \quad (3.20)$$

where H_{\perp} is a *centralizer* of $SU(2)_{\psi}$ in G . The cosets (3.20) for various groups G are of dimension $4(\tilde{h}_G - 2)$, and they are all classified [12, 13]. The *non-symmetric* spaces $(G/H_{\perp}) \otimes U(1)$ of dimension $4(\tilde{h}_G - 1)$ are also quaternionic. Therefore, the both different sets of cosets,

$$\frac{G}{H_{\perp} \otimes SU(2)_{\psi}} , \quad \text{and} \quad \frac{G \otimes U(1)}{H_{\perp}} , \quad (3.21)$$

support the non-linear $N = 4$ QSCA, but only the second one supports the ‘large’ linear $N = 4$ SCA too [5]. The list of compact Wolf spaces and the QSCA central charges of the associated $N = 4$ SCFT's are collected in Table 1. The only known *non-compact* quaternionic spaces are just non-compact analogues of those listed in Table 1, as well as some additional non-symmetric spaces found by Alekseevskii [13].

Table 1. The Wolf spaces, and the (Virasoro) central charges of the associated $N = 4$ SCFT's, with respect to the $N = 4 \hat{D}(2, 1; \alpha)$ QSCA. Here $k^+ = k_G \equiv \hat{k}$, $k^- = \tilde{h}_G - 2$, and $c_{\text{GS}} = 6(\hat{k} + 1)(\tilde{h}_G - 1)/(\hat{k} + \tilde{h}_G) - 3$.

$G/[H_\perp \otimes SU(2)]$		dim	\tilde{h}_G	c_{GS}
$\frac{Sp(n)}{Sp(n-1) \otimes Sp(1)}$	$n > 1$	$4n - 4$	$n + 1$	$6n - 3 - 6n^2/(\hat{k} + n + 1)$
$\frac{SU(n)}{SU(n-2) \otimes SU(2) \otimes U(1)}$	$n > 2$	$4n - 8$	n	$3(2n - 3) - 6(n - 1)^2/(\hat{k} + n)$
$\frac{SO(n)}{SO(n-4) \otimes SO(4)}$	$n > 4$	$4n - 16$	$n - 2$	$3(2n - 7) - 6(n - 3)^2/(\hat{k} + n - 2)$
$\frac{G_2}{SO(4)}$		8	4	$9 - 36/(\hat{k} + 4)$
$\frac{F_4}{Sp(3) \otimes Sp(1)}$		28	9	$45 - 384/(\hat{k} + 9)$
$\frac{E_6}{SU(6) \otimes SU(2)}$		40	12	$63 - 726/(\hat{k} + 12)$
$\frac{E_7}{SO(12) \otimes SU(2)}$		64	18	$99 - 1734/(\hat{k} + 18)$
$\frac{E_8}{E_7 \otimes SU(2)}$		112	30	$177 - 5220/(\hat{k} + 30)$

The ‘*small*’ linear $N = 4$ SCA can be formally obtained from the ‘*large*’ linear $N = 4$ SCA in the limit $k^- \rightarrow \infty$ and $k^+ \rightarrow 0$. We are however not in a position to get SCFT's based on the ‘*small*’ linear $N = 4$ SCA from our $N = 4$ coset construction since k^+ is the *only* parameter at our disposal according to eq. (3.16), which is not enough. This simple observation already makes a difference between the ‘*old*’ $N = 4$ strings [1], based on the ‘*small*’ linear $N = 4$ SCA, and the ‘*new*’ $N = 4$ strings based on the non-linear $N = 4 \hat{D}(2, 1; \alpha)$ QSCA [11].

The unitary highest-weight (positive energy) representations of the non-linear algebra were investigated by Günaydin, Petersen, Taormina and van Proeyen [14]. They showed that the central charge values leading to the *rational* $N = 4$ SCFT's (with *finite* numbers of different unitary representations) arise when $k^- = 0$, for the so-called *massless* representations labeled by the integer k_G and the half-integral highest-weight of the $su(2)$ subalgebra [14]. This implies $\tilde{h}_G = 2$ in the coset approach above. According to Table 1, no such unitary (massless) rational $N = 4$ SCFT's can appear in our construction.

4 $N = 4$ NLSM and WZNW

In the previous section, we constructed the $N = 4$ coset models by using the techniques of 2d CFT. A natural question arises whether our models can be identified with certain 2d *non-linear sigma-models* (NLSM's). The CFT construction applies to the holomorphic sector of a 2d field theory which corresponds to its left-moving degrees of freedom after the (inverse) Wick rotation. Therefore, by $N = 4$ supersymmetry above we actually mean $(4, 0)$ supersymmetry.⁹ In this section, we want to compare the $N = 4$ SCFT construction with the standard two-dimensional $N = 4$ NLSM construction known in the literature (see ref. [15] for a recent review), and build the relevant $N = 4$ WZNW actions on Wolf spaces.

4.1 $(4, 0)$ NLSM from the viewpoint of $(1, 0)$ superspace

Since an arbitrary bosonic NLSM can be made supersymmetric with respect to $N = 1$ or $(1, 0)$ supersymmetry, it seems to be quite natural to require an *explicit* $(1, 0)$ supersymmetry of the $(4, 0)$ supersymmetric NLSM in question. By ‘explicit’ we mean ‘off-shell’, in order to use superspace. It should be noticed however that only *on-shell* supersymmetry is required in SCFT. Since our $N = 4$ supersymmetry is going to be non-linearly realized in general, the standard (or harmonic) $N = 4$ superspace cannot be applied, at least naively, because it implies a *linearly* realized $N = 4$ supersymmetry, which is too restrictive for our purposes, as we already know from the previous section. To make contact with the standard results, we start from the $N = 1$ or $(1, 0)$ supersymmetric 2d NLSM.

The $(1, 0)$ superspace action for the most general $(1, 0)$ NLSM reads [15]¹⁰

$$I = \int d^2z d\theta^+ \left\{ (h_{ij} + b_{ij}) D_+ \Phi^i \partial_- \Phi^j + i h_{ab} \Psi_-^a \nabla_+ \Psi_-^b \right\}, \quad (4.1)$$

in terms of the $(1, 0)$ scalar superfields $\Phi^i(z^{\pm,=}, \theta^+)$ taking their values in a D -dimensional target manifold \mathcal{M} , and the $(1, 0)$ spinor superfields $\Psi_-^a(z^{\pm,=}, \theta^+)$ in a vector bundle \mathcal{K} over \mathcal{M} . In eq. (4.1), $D_+ = \frac{\partial}{\partial \theta^+} + i\theta^+ \partial_+$ denotes the flat $(1, 0)$ superspace covariant derivative, $\nabla_+ \Psi_-^a = D_+ \Psi_-^a + D_+ \Phi^i \Omega_i^a{}_b \Psi_-^b$ is its NLSM covariant generalization for the

⁹In two dimensions, a supersymmetry algebra can have p left-moving and q right-moving real supercharges.

¹⁰Our notation in this subsection is mostly self-explained, and it is *different* from the one used in the bulk of the paper.

spinor superfields, $h_{ij}(\Phi)$ is a metric on \mathcal{M} , $b_{ij}(\Phi)$ is an antisymmetric tensor on \mathcal{M} , $h_{ab}(\Phi)$ and $\Omega_i^a{}_b(\Phi)$ are a metric and a connection on the fibre \mathcal{K} , respectively. It is therefore assumed that \mathcal{M} must be a Riemannian manifold. In components, the action (4.2) takes the form

$$I = \int d^2z \left\{ (h_{ij} + b_{ij}) \partial_+ \Phi^i \partial_- \Phi^j + i h_{ij} \Lambda_+^i \left(\partial_- \Lambda_+^j + \Gamma_{kl}^j \partial_- \Phi^k \Lambda_+^l \right) \right. \\ \left. - i h_{ab} \Psi_-^a \left(\partial_+ \Psi_-^b + \partial_+ \Phi^i \Omega_i^b{}_c \Psi_-^c \right) - \frac{1}{2} F_{ijab} \Psi_-^a \Psi_-^b \Lambda_+^i \Lambda_+^j + h_{ab} F^a F^b \right\}, \quad (4.2)$$

where

$$\Phi^i = \Phi^i|, \quad \Lambda_+^i = D_+ \Phi^i|, \quad \Psi_-^a = \Psi_-^a|, \quad F^a = \nabla_+ \Psi_-^a|, \quad (4.3)$$

and $|$ denotes the leading component of a superfield. In eq. (4.2) the target space connection,

$$\Gamma_{jk}^i = \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} + B_{jk}^i, \quad B_{ijk} = \frac{3}{2} \partial_{[i} b_{jk]}, \quad (4.4)$$

and the fibre-valued curvature,

$$F_{ij}{}^a{}_b = \partial_i \Omega_j^a{}_b - \partial_j \Omega_i^a{}_b + \Omega_i^a{}_c \Omega_j^c{}_b - \Omega_j^a{}_c \Omega_i^c{}_b, \quad (4.5)$$

have been introduced. The scalars F^a are auxiliary, and they vanish on-shell.

The NLSM of eq. (4.1) has manifest off-shell $(1, 0)$ supersymmetry. Requiring further (non-manifest) supersymmetries implies certain restrictions on the NLSM couplings [9]. The form of additional supersymmetries is fixed by dimensional analysis:

$$\delta_\varepsilon \Phi^i = i \varepsilon_-^{(M)} h^{(M)i}{}_j(\Phi) D_+ \Phi^j, \\ \delta_\varepsilon \Psi_-^a = i \varepsilon_-^{(M)} h^{(M)a}{}_b(\Phi) \nabla_+ \Psi_-^b, \quad (4.6)$$

where some tensors $h^{(M)i}{}_j(\Phi)$ and $h^{(M)a}{}_b(\Phi)$ have been introduced, and $M = 1, 2, 3$ (*cf* eq. (3.4)). It should be noticed that the second line of eq. (4.6) is irrelevant on-shell where $\nabla_+ \Psi_-^b = 0$. The ‘canonical’ $(1, 0)$ supersymmetry can also be represented in the form (4.6) with $h^{(0)i}{}_j = \delta^i{}_j$ and $h^{(0)a}{}_b = \delta^a{}_b$, which again, as in the previous section, invites us to switch to the four-dimensional notation $\mu = (0, M)$.

Requiring the on-shell closure of the supersymmetry transformations (4.6) on the scalar superfields Φ^i alone results in *the same* conditions (3.6) and (3.13) appeared in the previous section, namely, (i) the existence of three independent complex structures satisfying the quaternionic algebra, and (ii) the vanishing Nijenhuis tensor! The on-shell closure on the spinor superfields Ψ_-^a yields

$$F_{ij}{}^a{}_b h^{\mu i}{}_{[k} h^{\nu j}{}_{l]} = \delta^{\mu\nu} F_{kl}{}^a{}_b, \quad (4.7)$$

in addition. Generally speaking, the conditions above are *not* enough to ensure the invariance of the action (4.1) with respect to the transformations (4.6), so that it could make a difference with the CFT approach. As is well known [9], the action (4.1) is actually invariant provided that, in addition, all the complex structures are hermitian *and* covariantly constant with respect to the connection (4.4),

$$\nabla_i h^\mu = 0 . \quad (4.8)$$

Therefore, the most general $N = 4$ supersymmetry conditions for the 2d NLSM's and the SCFT's defined on cosets are exactly the same! In geometrical terms, the $(2, 0)$ supersymmetry of the NLSM requires the holonomy of the connection (4.4) to be a subgroup of $U(D/2)$, and the vector bundle \mathcal{K} to be holomorphic [9, 15]. The $(4, 0)$ supersymmetry requires the holonomy to be a subgroup of $Sp(D/4) \otimes Sp(1)$, and the bundle \mathcal{K} to be holomorphic with respect to each complex structure. The latter is known to lead to hyper-kählerian ($b = 0$) or quaternionic ($b \neq 0$) manifolds, whose dimension is always a multiple of four. The holonomy conditions just mentioned easily follow from the vanishing commutator of the derivatives ∇_i on the complex structures h^μ , because of eq. (4.8).

4.2 An $N = 4$ gauged WZNW action for a Wolf space

The NLSM construction in the previous subsection is not explicit enough to accomodate the group-theoretical structure of the (S)CFT coset models. It is the gauged (super) WZNW actions that actually represent the relevant 2d field theories [16]. In ref. [6], Günaydin constructed the gauged $N = 4$ supersymmetric WZNW theories invariant under the ‘large’ linear $N = 4$ SCA. These gauged super WZNW theories are defined over $G \otimes U(1)$, and have the gauged subgroup H such that $G/[H \otimes SU(2)]$ is a Wolf space [6]. In this subsection, we modify the construction of ref. [6] to get the gauged super WZNW theories over the Wolf spaces. They are going to be invariant under the non-linear $N = 4$ QSCA $\hat{D}(2, 1; \alpha)$.

The standard WZNW action at level k is given by $kI(g)$, where

$$I(g) = -\frac{1}{4\pi} \int_{\Sigma} d^2z \operatorname{tr} \left(g^{-1} \partial g g^{-1} \bar{\partial} g \right) - \frac{1}{12\pi} \int_B d^3y \varepsilon^{\alpha\beta\gamma} \operatorname{tr} \left(g^{-1} \partial_{\alpha} g g^{-1} \partial_{\beta} g g^{-1} \partial_{\gamma} g \right) , \quad (4.9)$$

where $\partial B = \Sigma$, $\partial = \partial_z$, $\bar{\partial} = \partial_{\bar{z}}$, and the field $g(z, \bar{z})$ takes values in the group G .

The gauged WZNW action reads

$$I(g, A) = I(g) + \frac{1}{2\pi} \int_{\Sigma} d^2z \operatorname{tr} \left(A_z \bar{\partial} g g^{-1} - A_{\bar{z}} g^{-1} \partial g + A_z g A_{\bar{z}} g^{-1} - A_z A_{\bar{z}} \right) , \quad (4.10)$$

where the gauge fields $(A_z, A_{\bar{z}})$, taking their values in the Lie algebra \mathcal{H} of a diagonal subgroup H of the global $G_L \otimes G_R$ symmetry of the WZNW action (4.9), have been introduced.

The gauged $(1, 0)$ supersymmetric WZNW action for a coset G/H takes the form [16, 17]

$$I(g, A, \Psi) = I(g, A) + \frac{i}{4\pi} \int_{\Sigma} d^2z \operatorname{tr} \left(\Psi \bar{D} \Psi \right) , \quad (4.11)$$

where the 2d *Majorana-Weyl* (MW) fermions $\Psi^{\bar{a}}$ valued in the orthogonal complement \mathcal{N} of the Lie algebra \mathcal{H} in the Lie algebra \mathcal{G} have been introduced, $\bar{D}\Psi^{\bar{a}} = \bar{\partial}\Psi^{\bar{a}} + f^{\bar{a}bd}\Psi^{\bar{b}}A_{\bar{z}}^d$. Compared to the most general $(1, 0)$ NLSM in eq. (4.2), the $(1, 0)$ WZNW action (4.11) does not contain $(1, 0)$ spinor multiplets and has no quartic fermionic couplings.

The gauge transformations of the fields are

$$\delta g = [u, g] , \quad \delta A_z = Du , \quad \delta A_{\bar{z}} = \bar{D}u , \quad \delta \Psi = [u, \Psi] , \quad (4.12)$$

where $Du = \partial u - [A_z, u]$, $\bar{D}u = \bar{\partial}u - [A_{\bar{z}}, u]$, and u is the \mathcal{H} -valued infinitesimal gauge parameter. The on-shell $(1, 0)$ supersymmetry of the action (4.11) is

$$\delta g = i\varepsilon g \Psi , \quad \delta \Psi = \varepsilon \left(g^{-1} D_z g - i\Psi^2 \right)_{\mathcal{N}} , \quad \delta A = 0 . \quad (4.13)$$

The action (4.11) is a good starting point to examine further supersymmetries. In particular, as was shown by Witten [18], that action admits $(2, 0)$ supersymmetry when the coset space is *kählerian*, the canonical example being provided by the grassmannian manifolds $SU(n+m)/[SU(m) \otimes SU(n) \otimes U(1)]$ [19]. A quantization of the action for kählerian cosets results in a subclass of the KS models (subsect. 2.4), namely, those of them which have $\operatorname{rank} G = \operatorname{rank} H$. According to our discussion in subsect. 2.4, the rest of non-kählerian but still $N = 2$ supersymmetric KS models corresponds to the cases when $G/H = K \otimes U(1)^{2n}$, $n = 1, 2, \dots$, where K is a kählerian coset. It is trivial to generalize Witten's construction of the $N = 2$ gauged WZNW actions to the other (non-kählerian) cases, since the factor $U(1)^{2n}$ is abelian and, therefore, it merely contributes a free supersymmetric action for n scalar $(2, 0)$ supermultiplets. Without loss of generality,

we can restrict ourselves to the case of $n = 0$ in our construction of the $N = 4$ actions, modulo adding a free action for some number of chiral scalar $(4, 0)$ supermultiplets.¹¹

To this end, we are going to elaborate the structure of the gauged super-WZNW theories on the Wolf spaces (3.20), by using Günaydin's results about coset realizations of the $N = 4$ extended SCA's over the so-called *Freudenthal triple systems* (FTS's) [6]. A convenient (Kantor) decomposition of the Lie algebra \mathcal{G} is given by its decomposition into the eigenspaces with respect to the grading operator H_ψ [20],

$$\mathcal{G} = \mathcal{G}^{(-2)} \oplus \mathcal{G}^{(-1)} \oplus \mathcal{G}^{(0)} \oplus \mathcal{G}^{(+1)} \oplus \mathcal{G}^{(+2)} , \quad (4.14)$$

where the H_ψ -eigenvalues appear as superscripts (in brackets).¹² The one-dimensional spaces $\mathcal{G}^{(-2)}$ and $\mathcal{G}^{(+2)}$ just comprise $E_{\psi-}$ and $E_{\psi+}$, respectively, whereas $\mathcal{G}^{(0)}$ can be identified with $\mathcal{H}_\perp \oplus H_\psi$, where \mathcal{H}_\perp is the Lie algebra of H_\perp . Let $E_{\bar{a}\pm}$ be the generators of $\mathcal{G}^{(\pm 1)}$, and $H_{\bar{a}\bar{c}}^\perp$ the generators of \mathcal{H}_\perp in the Cartan-Weyl-type basis. The non-trivial commutation relations of \mathcal{G} are then given by (the signs are correlated!)

$$\begin{aligned} [E_{\bar{a}\pm}, E_{\bar{c}\pm}] &= \Omega_{\bar{a}\bar{c}}^\pm E_{\psi\pm} , & [E_{\bar{a}+}, E_{\bar{c}-}] &= H_{\bar{a}\bar{c}}^\perp + \delta_{\bar{a}\bar{c}} H_\psi \equiv H_{\bar{a}\bar{c}} , \\ [E_{\psi\pm}, E_{\bar{a}\mp}] &= \Omega_{\bar{a}\bar{c}}^\mp E_{\bar{c}\pm} , & [H_{\bar{a}\bar{b}}, E_{\bar{c}\pm}] &= \pm f_{\bar{a}\bar{b}\bar{c}\bar{g}} E_{\bar{g}\pm} , \\ [H_{\bar{a}\bar{b}}, H_{\bar{c}\bar{d}}] &= f_{\bar{a}\bar{b}\bar{g}\bar{c}} H_{\bar{g}\bar{d}} - f_{\bar{a}\bar{b}\bar{d}\bar{g}} H_{\bar{c}\bar{g}} . \end{aligned} \quad (4.15)$$

Here $f_{\bar{a}\bar{b}\bar{c}\bar{d}}$ are the structure constants of $\mathcal{H}_\perp \oplus H_\psi$, whose (Cartan-Weyl) normalization is fixed by the conditions [6]

$$f_{\bar{a}\bar{a}\bar{c}\bar{d}} = (\tilde{h}_G - 2) \delta_{\bar{c}\bar{d}} , \quad f_{\bar{a}\bar{b}\bar{b}\bar{c}} = (\tilde{h}_G - 1) \delta_{\bar{a}\bar{c}} , \quad (4.16)$$

and which satisfy the identity

$$f_{\bar{a}\bar{b}\bar{c}\bar{d}} - f_{\bar{a}\bar{c}\bar{b}\bar{d}} = \Omega_{\bar{a}\bar{d}}^+ \Omega_{\bar{b}\bar{c}}^- . \quad (4.17)$$

The matrix $\Omega_{\bar{a}\bar{c}}^\pm$ introduced in eq. (4.15) represents a natural *symplectic* structure associated with a Wolf space [21],

$$(\Omega^\pm)^{-1} = \Omega^\mp , \quad (\Omega^\pm)^T = -\Omega^\pm . \quad (4.18)$$

¹¹Free chiral scalar $N = 4$ supermultiplets are still relevant in $N = 4$ string theory, since they contribute to the conformal anomaly. They play the role similar to free scalars appearing in the toroidally compactified (four-dimensional) superstrings.

¹² The elements of $\mathcal{G}^{(-1)}$ can be put in one-to-one correspondence with FTS, the latter being usually represented by a division algebra [21].

Under hermitian conjugation, one has $(E_{\psi\pm})^\dagger = E_{\psi\mp}$, $(E_{\bar{a}\pm})^\dagger = E_{\bar{a}\mp}$ and $(\Omega^\pm)^\dagger = -\Omega^\mp$.

It is straightforward to write down the defining OPE's of the affine Lie algebra $\hat{\mathcal{G}}$, in the form adapted to the commutation relations (4.15), namely

$$\begin{aligned}
E_{\bar{a}\pm}(z)E_{\bar{c}\pm}(w) &\sim \frac{\Omega_{\bar{a}\bar{c}}^\pm E_{\psi\pm}(w)}{z-w}, & E_{\bar{a}+}(z)E_{\bar{c}-}(w) &\sim \frac{k_G \delta_{\bar{a}\bar{c}}}{(z-w)^2} + \frac{H_{\bar{a}\bar{c}}(w)}{z-w}, \\
E_{\psi\pm}(z)E_{\bar{a}\mp}(w) &\sim \frac{\Omega_{\bar{a}\bar{c}}^\mp E_{\bar{c}\pm}(w)}{z-w}, & H_{\bar{a}\bar{b}}(z)E_{\bar{c}\pm}(w) &\sim \pm \frac{f_{\bar{a}\bar{b}\bar{c}\bar{g}} E_{\bar{g}\pm}(w)}{z-w}, \\
H_{\bar{a}\bar{b}}(z)H_{\bar{c}\bar{d}}(w) &\sim \frac{k_G f_{\bar{a}\bar{b}\bar{c}\bar{d}}}{(z-w)^2} + \frac{1}{z-w} \left[f_{\bar{a}\bar{b}\bar{g}\bar{c}} H_{\bar{g}\bar{d}}(w) - f_{\bar{a}\bar{b}\bar{d}\bar{g}} H_{\bar{c}\bar{g}}(w) \right], \\
E_{\psi+}(z)E_{\psi-}(w) &\sim \frac{k_G}{(z-w)^2} + \frac{2H_\psi(w)}{z-w}, & H_\psi(z)E_{\psi\pm}(w) &\sim \frac{\pm E_\psi(w)}{z-w}. \quad (4.19)
\end{aligned}$$

The gauged (4, 0) supersymmetric WZNW action on a Wolf space (3.20) is given by eq. (4.11), where the gauged group H has to be $H_\perp \otimes SU(2)_\psi$, and free MW fermions Ψ should be assigned only for the FTS generators of $\mathcal{G}^{(-1)} \oplus \mathcal{G}^{(+1)}$, i.e. for $E_{\bar{a}\pm}$. The corresponding on-shell (holomorphic) fermions, $\psi^{\bar{a}\pm}(z)$, satisfy the canonical OPE

$$\psi^{\bar{a}+}(z)\psi^{\bar{b}-}(w) \sim -\frac{\delta^{\bar{a}\bar{b}}}{z-w}. \quad (4.20)$$

The generators of the non-linear $\hat{D}(2, 1; \alpha)$ QSCA in the $N = 4$ gauged WZNW theory were identified in ref. [6]. Compared with eq. (3.18) in the SCFT approach, the $N = 4$ supersymmetry generators in the field theory (WZNW) approach naturally appear in the (2, 2) representation instead of the (1, 3) one in eq. (3.18), namely [6]

$$\begin{aligned}
G^0(z) \pm iG^1(z) &= \frac{2}{\sqrt{k_G + \tilde{h}_G}} \psi^{\bar{a}\pm}(z) E_{\bar{a}\pm}(z), \\
G^2(z) \pm iG^3(z) &= \frac{\mp 2}{\sqrt{k_G + \tilde{h}_G}} \psi^{\bar{a}\pm}(z) \Omega_{\bar{a}\bar{c}}^\mp E_{\bar{c}\pm}(z). \quad (4.21)
\end{aligned}$$

The generators of the first $\widehat{su(2)}$ affine subalgebra (at level k_G) of the QSCA are just given by the $SU(2)_\psi$ currents $E_{\psi\pm}(z)$ and $H_\psi(z)$ – see the last two lines of eq. (4.19). The generators of the second $\widehat{su(2)}$ affine subalgebra at level \tilde{h}_G are bilinears of free fermions [6],

$$J_\pm(z) = \frac{1}{2} \Omega_{\bar{a}\bar{c}}^\pm \psi^{\bar{a}\mp}(z) \psi^{\bar{c}\mp}(z), \quad J_3(z) = -\frac{1}{2} \psi^{\bar{a}+}(z) \psi^{\bar{a}-}(z). \quad (4.22)$$

Finally, the QSCA stress tensor reads [6]

$$T = \frac{1}{k_G + \tilde{h}_G} \left\{ \frac{1}{2} (E_{\bar{a}+} E_{\bar{a}-} + E_{\bar{a}-} E_{\bar{a}+}) + \frac{1}{2} (E_{\psi+} E_{\psi-} + E_{\psi-} E_{\psi+}) + H_\psi^2 \right. \\ \left. + \frac{k_G + 1}{2} (\psi^{\bar{a}+} \partial \psi^{\bar{a}-} + \psi^{\bar{a}-} \partial \psi^{\bar{a}+}) - H_{\bar{a}\bar{c}}^\perp \psi^{\bar{a}+} \psi^{\bar{a}-} + \frac{1}{2} \Omega_{\bar{a}\bar{b}}^+ \Omega_{\bar{c}\bar{d}}^- \psi^{\bar{a}-} \psi^{\bar{b}-} \psi^{\bar{c}+} \psi^{\bar{d}+} \right\} . \quad (4.23)$$

It is instructive to compare the $N = 4$ QSCA generators obtained from the $N = 4$ SCFT coset approach in eq. (3.18), with the $N = 4$ WZNW generators given above. First, we immediately see that they actually coincide after identifying

$$E_{\bar{a}\pm}(z) = \frac{i}{\sqrt{2k}} \hat{J}_{\bar{a}\pm}(z) , \quad \text{where} \quad k = k_G + \tilde{h}_G , \quad (4.24)$$

and using the crucial identity (4.17). Second, after identifying the generators as above, we find the quaternionic structure $\{h^\mu\}$, $\mu = 0, 1, 2, 3$, on a Wolf space, in terms of the symplectic structure of the associated FTS. The first complex structure takes, of course, the canonical form, as it should,

$$h_{(\bar{a}\pm)(\bar{b}\pm)}^{(1)} = \begin{pmatrix} -i\delta_{\bar{a}\bar{b}} & 0 \\ 0 & +i\delta_{\bar{a}\bar{b}} \end{pmatrix} . \quad (4.25)$$

As far as the other two complex structures are concerned, we find

$$h_{(\bar{a}\pm)(\bar{b}\pm)}^{(2)} = \begin{pmatrix} -\Omega_{\bar{a}\bar{b}}^- & 0 \\ 0 & +\Omega_{\bar{a}\bar{b}}^+ \end{pmatrix} , \quad h_{(\bar{a}\pm)(\bar{b}\pm)}^{(3)} = \begin{pmatrix} +i\Omega_{\bar{a}\bar{b}}^- & 0 \\ 0 & = i\Omega_{\bar{a}\bar{b}}^+ \end{pmatrix} . \quad (4.26)$$

In particular, $h^{(1)} h^{(2)} = h^{(3)}$, as it should. The dimension of a Wolf space, $D_W = 4(\tilde{h}_G - 2)$, is clearly twice the dimension of the corresponding FTS.

Summarizing the above-mentioned in this section, the $N = 4$ field theory (WZNW) approach leads to *the same* results as the $N = 4$ SCFT approach, although in a more tedious way.

5 New $N = 4$ strings

We are now in a position to discuss $N = 4$ strings propagating on Wolf spaces. The coset realizations of the $N = 4$ QSCA considered above give relevant constraints on the $N = 4$ string physical states in the form

$$\left\{ \frac{1}{2} (E_{\bar{a}+} E_{\bar{a}-} + E_{\bar{a}-} E_{\bar{a}+}) + \frac{1}{2} (k_G + 1) (\psi^{\bar{a}+} \partial \psi^{\bar{a}-} + \psi^{\bar{a}-} \partial \psi^{\bar{a}+}) \right. \\ \left. - H_{\bar{a}\bar{c}}^\perp \psi^{\bar{a}+} \psi^{\bar{a}-} + \frac{1}{2} \Omega_{\bar{a}\bar{b}}^+ \Omega_{\bar{c}\bar{d}}^- \psi^{\bar{a}-} \psi^{\bar{b}-} \psi^{\bar{c}+} \psi^{\bar{d}+} \right\} |\text{phys}\rangle = 0 ,$$

$$\begin{aligned}
\psi^{\bar{a}\pm} E_{\bar{a}\pm} |\text{phys}\rangle &= \psi^{\bar{a}\pm} \Omega_{\bar{a}\bar{c}}^{\mp} E_{\bar{c}\pm} |\text{phys}\rangle = 0 , \\
E_{\psi\pm} |\text{phys}\rangle &= H_{\psi} |\text{phys}\rangle = 0 , \\
\Omega_{\bar{a}\bar{c}}^{\pm} \psi^{\bar{a}\mp} \psi^{\bar{c}\mp} |\text{phys}\rangle &= \psi^{\bar{a}+} \psi^{\bar{a}-} |\text{phys}\rangle = 0 ,
\end{aligned} \tag{5.1}$$

where eqs. (4.21), (4.22) and (4.23) have been used. It is obvious that these constraints are very different from the ones proposed in ref. [1], and, therefore, they define a new theory of $N = 4$ strings. Note, in particular, a presence of the quartic fermionic term in the second line of eq. (5.1). Although the string constraints (5.1) look very complicated, the $N = 4$ QSCA they satisfy actually allows us to get information about their content from the corresponding $N = 4$ SCFT.

The full invariant 2d action for this $N = 4$ string theory is obtained by promoting the superconformal symmetries of the $N = 4$ gauged WZNW action to the local level. As is usual in string theory, the string constraints (5.1) are to be in one-to-one correspondence with proper on-shell $N = 4$ supergravity fields. In our case, the new W -type $N = 4$ supergravity seems to be needed [11], and its gauge fields are ¹³

$$e_{\alpha}^a , \quad \chi_{\alpha}^{\mu} , \quad B_{\alpha}^{I\pm} , \tag{5.2}$$

where e_{α}^a is a zweibein, χ_{α}^{μ} are four 2d MW gravitinos, and $B_{\alpha}^{I\pm}$ are six $SU(2) \otimes SU(2)$ gauge fields. The full action is obtained by adding to the rigid $N = 4$ action (4.11) the Noether coupling for the $N = 4$ supersymmetry, and minimally covariantizing the result with respect to all the gauge fields in eq. (5.2) [11]. No additional terms are needed in the action. ¹⁴ Like in the ‘old’ invariant $N = 4$ string action found by Pernici and Nieuwenhuizen [8], the rigid and local $N = 4$ models have *the same* geometry for the internal NLSM manifold parametrized by the scalar fields (i.e. quaternionic), and no constraints on the $Sp(1)$ curvature of a quaternionic manifold arise, unlike in four dimensions [7]. Instead of concentrating on the action and the transformation laws [22], we proceed with the BRST quantization.

The gauge field content of the \tilde{D}_4 conformal 2d supergravity is balanced by the gauge symmetries as usual, which implies no off-shell degrees of freedom (up to moduli). In quantum theory, some of the gauge symmetries may become anomalous and thereby some of the gauge degrees of freedom may become physical.

The BRST ghosts appropriate for this case are:

¹³Following ref. [11], we call it \tilde{D}_4 supergravity.

¹⁴Of course, as is always the case in the Noether procedure, the transformation laws of the fields receive proper modifications.

- the conformal ghosts (b, c) , an anticommuting pair of world-sheet free fermions of conformal dimensions $(2, -1)$, respectively;
- the $N = 4$ superconformal ghosts (β^μ, γ^μ) of conformal dimensions $(\frac{3}{2}, -\frac{1}{2})$, respectively, in the fundamental (vector) representation of $SO(4)$;
- the $SU(2)_+ \otimes SU(2)_-$ internal symmetry ghosts $(\tilde{b}^{I\pm}, \tilde{c}^{I\pm})$ of conformal dimensions $(1, 0)$, respectively, in the adjoint representation of $SU(2)_\pm$.

The conformal ghosts

$$b(z) = \sum_{n \in \mathbf{Z}} b_n z^{-n-2}, \quad c(z) = \sum_{n \in \mathbf{Z}} c_n z^{-n+1}, \quad (5.3)$$

have the following OPE and anticommutation relations:

$$b(z) c(w) \sim \frac{1}{z-w}, \quad \{c_m, b_n\} = \delta_{m+n,0}. \quad (5.4)$$

The superconformal ghosts

$$\beta^\mu(z) = \sum_{r \in \mathbf{Z} + 1/2} \beta_r^\mu z^{-r-3/2}, \quad \gamma^\mu(z) = \sum_{r \in \mathbf{Z} + 1/2} \gamma_r^\mu z^{-r+1/2}, \quad (5.5)$$

satisfy

$$\beta^\mu(z) \gamma^\nu(w) \sim \frac{-\delta^{\mu\nu}}{z-w}, \quad [\gamma_r^\mu, \beta_s^\nu] = \delta^{\mu\nu} \delta_{r+s,0}. \quad (5.6)$$

An integer or half-integer moding of these generators corresponds to the usual distinction between the Ramond- and Neveu-Schwarz-type sectors. Finally, the fermionic $SU(2)_\pm$ ghosts

$$\tilde{b}^{I\pm}(z) = \sum_{n \in \mathbf{Z}} \tilde{b}_n^{I\pm} z^{-n-1}, \quad \tilde{c}^{I\pm}(z) = \sum_{n \in \mathbf{Z}} \tilde{c}_n^{I\pm} z^{-n}, \quad (5.7)$$

have

$$\tilde{b}^{I\pm}(z) \tilde{c}^{J\pm}(w) \sim \frac{\delta^{IJ}}{z-w}, \quad \{\tilde{c}_m^{I\pm}, \tilde{b}_n^{J\pm}\} = \delta^{IJ} \delta_{m+n,0}. \quad (5.8)$$

The BRST charge $Q_{\text{BRST}} = \oint_0 \frac{dz}{2\pi i} j_{\text{BRST}}(z)$ was calculated in ref. [11]. The BRST current $j_{\text{BRST}}(z)$ takes the form (modulo total derivative)

$$\begin{aligned} j_{\text{BRST}}(z) = & cT + \gamma^\mu G^\mu + \tilde{c}^{I\pm} J^{I\pm} + bc\partial c - c\tilde{b}^{I\pm} \partial \tilde{c}^{I\pm} - \frac{1}{2} c \gamma^\mu \partial \beta^\mu - \frac{3}{2} c \beta^\mu \partial \gamma^\mu - b \gamma^\mu \gamma^\mu \\ & - \frac{1}{2} \tilde{c}^{I\pm} (t^{I\pm})^{\mu\nu} \beta^\mu \gamma^\nu + \left[\tilde{b}^{I+} (t^{I+})^{\mu\nu} + \tilde{b}^{I-} (t^{I-})^{\mu\nu} \right] (\gamma^\mu \partial \gamma^\nu - \gamma^\nu \partial \gamma^\mu) \\ & - \frac{1}{2} \varepsilon^{IJK} \tilde{c}^I + \tilde{c}^J + \tilde{b}^{K+} - \frac{1}{2} \varepsilon^{IJK} \tilde{c}^I - \tilde{c}^J - \tilde{b}^{K-} + \frac{1}{2} \Lambda_{(I\pm)(J\pm)}^{\mu\nu} J^{(I\pm)} \tilde{b}^{(J\pm)} \gamma^\mu \gamma^\nu \\ & - \frac{1}{24} \Lambda_{(I\pm)(J\pm)}^{\mu\nu} \Lambda_{K(\pm)L(\pm)}^{\lambda\rho} \varepsilon^{IKN} \tilde{b}^{J(\pm)} \tilde{b}^{L(\pm)} (\tilde{b}^{N+} + \tilde{b}^{N-}) \gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\rho, \end{aligned} \quad (5.9)$$

where the constant ‘non-linearity’ tensor Λ can be easily read off from the last term on the r.h.s. of the supersymmetry algebra (A.1c) after rewriting it in terms of the self-dual currents defined by eq. (3.2).

The quantum BRST charge (5.9) is nilpotent if and only if [11]

$$k^+ = k^- = -2 , \quad (5.10)$$

which implies, in particular

$$c_{\text{tot}} \equiv c_{\text{matter}} + c_{\text{gh}} = \left[\frac{6(k^+ + 1)(k^- + 1)}{k^+ + k^- + 2} - 3 \right] + 6 = 0 . \quad (5.11)$$

In calculating the ghost contributions to the central charge, we used the standard formula of conformal field theory [23],

$$\begin{aligned} c_{\text{gh}} &= 2 \sum_{\lambda} n_{\lambda} (-1)^{2\lambda+1} (6\lambda^2 - 6\lambda + 1) \\ &= 1 \times (-26) + 4 \times (+11) + \frac{1}{2} 4(4 - 1) \times (-2) = +6 , \end{aligned} \quad (5.12)$$

where λ is the conformal dimension and n_{λ} is the number of the conjugated ghost pairs: $\lambda = 2, 3/2, 1$ and $n_{\lambda} = 1, 4, 6$, respectively.

To cancel the positive ghost contribution, we need therefore the negative central charge (-6) for a matter representation. According to Table I, the level k_G is also negative for a negative central charge. This simple observation already excludes unitary representations of the $N = 4$ QSCA, and, hence, the physical space defined by the constraints (5.1) has little chance to be positive definite. Moreover, comparing eqs. (3.16) and (5.10) in the case of a Wolf space to be used as the background for the $N = 4$ string propagation, we conclude that $\tilde{h}_G = 0$. Therefore, the group G has to be abelian. It leaves us only (locally flat) tori as the consistent $N = 4$ string backgrounds.

6 Conclusion and Outlook

Our main results are given by the title and the abstract. Contrary to the conventional approach to $N = 4$ strings based on the ‘small’ $N = 4$ SCA, we used the non-linear $N = 4$ supersymmetric QSCA, which is more general. We generalized the supersymmetric coset construction to that $N = 4$ case, constructed the relevant $N = 4$ gauged WZNW actions, and defined the BRST quantized theory of $N = 4$ strings propagating on the

Wolf spaces. Due to the non-linearity of the underlying gauged algebra, it is not possible to build new representations by ‘tensoring’ the known ones, similarly to representations of W algebras. Still, even that rather general framework didn’t save us from the disaster: the Wolf spaces as the $N = 4$ string backgrounds are forbidden by the quantum BRST charge nilpotency conditions, as we showed. The only spaces allowed are just tori, which are locally flat. The result is rather surprising since the Wolf spaces naturally appear as solutions in the $N = 4$ coset construction. Consistent backgrounds for the $N = 4$ string propagation may also exist outside cosets.

To this end, we would like to comment on the issue of *off-shell* extensions of the $N = 4$ gauged WZNW actions. All our considerations above were merely *on-shell*, which was important in our general analysis. In particular, the super WZNW theories on the Wolf spaces are only invariant under the on-shell $N = 4$ supersymmetry which is given by the on-shell current algebra, and which is non-linearly realised. In terms of the transformation laws for the super WZNW fields, the non-linearity implies certain field dependence of the ‘structure constants’ in the commutator of two $N = 4$ supertransformations. In order to get an off-shell description if any, it is the necessary first step that the $N = 4$ supersymmetry should be linearized. It has been known for some time [2, 5, 10] that it is indeed possible, although not for the super WZNW theories on the Wolf spaces W , but for those on cosets of the type $W \otimes SU(2) \otimes U(1)$ (*cf* eq. (3.21)), where the additional fields belonging to the $SU(2) \otimes U(1)$ group factor serve as the ‘auxiliaries’ to linearize the on-shell current algebra. Given the linear $N = 4$ supersymmetry, the natural way for an off-shell approach would be to use $N = 4$ superspace. However, it is *not* known how to formulate the $N = 4$ super WZNW theory on a non-trivial Wolf space in $N = 4$ superspace, even without coupling to any 2d supergravity theory [26]. The related problem recently discovered [27] is a variety of ways to define an on-shell $N = 4$ scalar supermultiplet, as well as its off-shell realizations, in two dimensions. The $N = 4$ superspace constraints for scalar supermultiplets are of most importance, since they simultaneously determine kinematics of the propagating fields. Clearly, there are still some unsolved problems around [22].

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Appendix: $\hat{D}(1, 2; \alpha)$ QSCA and ‘large’ $N = 4$ SCA

The non-trivial OPE’s of the $\hat{D}(1, 2; \alpha)$ QSCA are given by [2]

$$T^{\mu\nu}(z)G^\rho(w) \sim \frac{1}{z-w} [\delta^{\mu\rho}G^\nu(w) - \delta^{\nu\rho}G^\mu(w)] , \quad (A.1a)$$

$$\begin{aligned} J^{\mu\nu}(z)J^{\rho\lambda}(w) \sim & \frac{1}{z-w} [\delta^{\mu\rho}J^{\nu\lambda}(w) - \delta^{\nu\rho}J^{\mu\lambda}(w) + \delta^{\nu\lambda}J^{\mu\rho}(w) - \delta^{\mu\lambda}J^{\nu\rho}(w)] \\ & - \frac{1}{2}(k^+ + k^-) \frac{\delta^{\mu\rho}\delta^{\nu\lambda} - \delta^{\mu\lambda}\delta^{\nu\rho}}{(z-w)^2} - \frac{1}{2}(k^+ - k^-) \frac{\varepsilon^{\mu\nu\rho\lambda}}{(z-w)^2} , \end{aligned} \quad (A.1b)$$

$$\begin{aligned} G^\mu(z)G^\nu(w) \sim & \frac{4k^+k^-}{(k^+ + k^- + 2)} \frac{\delta^{\mu\nu}}{(z-w)^3} + \frac{2T(w)\delta^{\mu\nu}}{z-w} \\ & - \frac{k^+ + k^-}{k^+ + k^- + 2} \left[\frac{2J^{\mu\nu}(w)}{(z-w)^2} + \frac{\partial J^{\mu\nu}(w)}{z-w} \right] \\ & + \frac{k^+ - k^-}{k^+ + k^- + 2} \varepsilon^{\mu\nu\rho\lambda} \left[\frac{J^{\rho\lambda}(w)}{(z-w)^2} + \frac{\partial J^{\rho\lambda}(w)}{2(z-w)} \right] \\ & - \frac{\varepsilon^{\mu\rho\lambda\zeta}\varepsilon^{\nu\rho\tau\omega}}{2(k^+ + k^- + 2)} : J^{\lambda\zeta}J^{\tau\omega} : (w) . \end{aligned} \quad (A.1c)$$

The antisymmetric tensor $J^{\mu\nu}(z)$ in the adjoint of $SO(4)$ can be decomposed into its self-dual $SU(2)$ components, see eqs. (3.2) and (3.3).

The OPE’s describing the action of $J^{M\pm}(z)$ read

$$\begin{aligned} J^{M\pm}(z)J^{N\pm}(w) & \sim \frac{\varepsilon^{MNP}J^{P\pm}(w)}{z-w} + \frac{-k^\pm\delta^{MN}}{2(z-w)^2} , \\ J^{M\pm}(z)G^\mu(w) & \sim \frac{\frac{1}{2}(t^{M\pm})^{\mu\nu}G^\nu(w)}{z-w} , \end{aligned} \quad (A.2)$$

where *two* arbitrary ‘levels’ k^\pm for both independent $\widehat{su(2)}_\pm$ affine Lie algebra components have been introduced.

Though $\hat{D}(1, 2; \alpha)$ is a non-linear QSCA, it can be turned into a *linear* SCA by adding some ‘auxiliary’ fields, namely, four free fermions $\psi^\mu(z)$ of dimension 1/2, and a free bosonic current $U(z)$ of dimension 1 [2]. The new fields have canonical OPE’s,

$$\psi^\mu(z)\psi^\nu(w) \sim \frac{-\delta^{\mu\nu}}{z-w} , \quad U(z)U(w) \sim \frac{-1}{(z-w)^2} . \quad (A.3)$$

The fermionic fields $\psi^\mu(z)$ transform in $(2, 2)$ representation of $SU(2)_+ \otimes SU(2)_-$,

$$J^{M\pm}(z)\psi^\mu(w) \sim \frac{\frac{1}{2}(t^{M\pm})^{\mu\nu}\psi^\nu(w)}{z-w} , \quad (A.4)$$

whereas the singlet $U(1)$ -current $U(z)$ can be thought of as derivative of a free scalar boson, $U(z) = i\partial\phi(z)$. The new currents takes the form [2]

$$\begin{aligned}
T_{\text{tot}} &= T - \frac{1}{2} : U^2 : - \frac{1}{2} : \partial\psi^\mu\psi^\mu : , \\
G_{\text{tot}}^\mu &= G^\mu - U\psi^\mu + \frac{1}{3\sqrt{2(k^+ + k^- + 2)}} \varepsilon^{\mu\nu\rho\lambda} \psi^\nu \psi^\rho \psi^\lambda \\
&\quad - \sqrt{\frac{2}{k^+ + k^- + 2}} \psi^\nu \left[(t^{M+})^{\nu\mu} J^{M+} - (t^{M-})^{\nu\mu} J^{M-} \right] , \\
J_{\text{tot}}^{M\pm} &= J^{M\pm} + \frac{1}{4} (t^{M\pm})^{\mu\nu} \psi^\mu \psi^\nu ,
\end{aligned} \tag{A.5}$$

in terms of the initial $\hat{D}(1, 2; \alpha)$ QSCA currents T , G^μ and $J^{M\pm}$. It follows that the generators T_{tot} , G_{tot}^μ , $J_{\text{tot}}^{M\pm}$, ψ^μ and U have closed OPE's among themselves, and define a 'large' linear $N = 4$ SCA with $\widehat{su(2)} \oplus \widehat{su(2)} \oplus \widehat{u(1)}$ affine Lie subalgebra [2]. The non-trivial OPE's of the 'large' linear $N = 4$ SCA are

$$\begin{aligned}
T_{\text{tot}}(z)T_{\text{tot}}(w) &\sim \frac{\frac{1}{2}(c+3)}{(z-w)^4} + \frac{2T_{\text{tot}}(w)}{(z-w)^2} + \frac{\partial T_{\text{tot}}(w)}{z-w} , \\
T_{\text{tot}}(z)\mathcal{O}(w) &\sim \frac{h_{\mathcal{O}}\mathcal{O}(w)}{(z-w)^2} + \frac{\partial\mathcal{O}(w)}{z-w} , \\
J_{\text{tot}}^{M\pm}(z)J_{\text{tot}}^{N\pm}(w) &\sim \frac{\varepsilon^{MNP}J_{\text{tot}}^{P\pm}(w)}{z-w} - \frac{(k^\pm + 1)\delta^{MN}}{2(z-w)^2} , \\
J_{\text{tot}}^{M\pm}(z)G_{\text{tot}}^\mu(w) &\sim \frac{\frac{1}{2}(t^{M\pm})^{\mu\nu}G_{\text{tot}}^\nu(w)}{z-w} \mp \frac{k^\pm + 1}{\sqrt{2(k^+ + k^- + 2)}} \frac{(t^{M\pm})^{\mu\nu}\psi^\nu(w)}{(z-w)^2} , \\
G_{\text{tot}}^\mu(z)G_{\text{tot}}^\nu(w) &\sim \frac{\frac{2}{3}(c+3)\delta^{\mu\nu}}{(z-w)^3} + \frac{2T_{\text{tot}}(w)\delta^{\mu\nu}}{z-w} - \frac{2}{k^+ + k^- + 2} \left[\frac{2}{(z-w)^2} + \frac{1}{z-w}\partial_w \right] \\
&\quad \times \left[(k^- + 1)(t^{M+})^{\mu\nu}J_{\text{tot}}^{M+}(w) + (k^+ + 1)(t^{M-})^{\mu\nu}J_{\text{tot}}^{M-}(w) \right] , \\
\psi^\mu(z)G_{\text{tot}}^\nu(w) &\sim \frac{1}{z-w} \sqrt{\frac{2}{k^+ + k^- + 2}} \left[(t^{M+})^{\mu\nu}J_{\text{tot}}^{M+}(w) - (t^{M-})^{\mu\nu}J_{\text{tot}}^{M-}(w) \right] \\
&\quad + \frac{U(w)\delta^{\mu\nu}}{z-w} , \\
U(z)G_{\text{tot}}^\mu(w) &\sim \frac{\psi^\mu(w)}{(z-w)^2} ,
\end{aligned} \tag{A.6}$$

where \mathcal{O} stands for the generators G_{tot} , J_{tot} and ψ of dimension $3/2$, 1 and $1/2$. The

$\hat{D}(1, 2; \alpha)$ QSCA central charge is

$$c = \frac{6(k^+ + 1)(k^- + 1)}{k^+ + k^- + 2} - 3. \quad (A.7)$$

We define the α -parameter of the $\hat{D}(1, 2; \alpha)$ QSCA as a ratio of its two affine ‘levels’, $\alpha \equiv k^-/k^+$, which measures the relative asymmetry between the two $\widehat{su}(2)$ affine Lie algebras. When $\alpha = 1$, i.e. $k^- = k^+ \equiv k$, the $\hat{D}(1, 2; 1)$ QSCA coincides with the $SO(4)$ Bershadsky-Knizhnik QSCA [24, 25]. The ‘levels’ and the central charges of those QSCA’s are different, $k_{\text{large}}^\pm = k^\pm + 1$ and $c_{\text{large}} = c + 3$. The exceptional ‘small’ $N = 4$ SCA with the $\widehat{su}(2)$ affine Lie algebra component [1] follows from the ‘large’ $N = 4$ SCA in the limit $\alpha \rightarrow \infty$ or $\alpha \rightarrow 0$, where either $k^- \rightarrow \infty$ or $k^+ \rightarrow \infty$, respectively. Taking the limit results in the central charge $c_{\text{small}} = 6k$, where k is an arbitrary ‘level’ of the remaining $\widehat{su}(2)$ component.

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